Simple Harmonic Motion

INTRODUCTION

A particle has oscillatory (vibrational) motion when it moves periodically about a stable equilibrium position. The motion of a pendulum is oscillatory. A weight attached to a stretched spring, once it is released, starts oscillating.Of all the oscillatory motions, most important is called simple harmonic motion (S.H.M.). In this type of oscillatory motion, displacement, velocity, acceleration and force all vary (w.r.t to time) in a way that can be described by either the sine or the cosine function collectively called sinusoids.

In S.H.M. the restoring force acting on the particle is directly proportional to its displacment from the equilibrium position.

$$F \propto x$$

$$F = -Kx$$

K is the constant of proportionality and (–ve) sign shows that the force is always directed toward the mean position.

Using newtons second law SHM is dessembed by,

$$F = ma = \frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where, $\omega = \sqrt{\frac{k}{m}}$ = Angular frequency of SHM.

KINEMATICS OF S.H.M.

A particle has S.H.M. along the axis OX when its displacement x relative to the origin of coordinate system is given as a function of time by the relation.

$$x = A\sin(wt + \phi)$$

The quantity $(wt + \phi)$ is called the phase angle of the SHM and ϕ is called the initial phase i.e. phase at $t = 0$.

The maximum displacement from the origin A, is called the amplitude of the SHM.

Period : $T = \frac{2\pi}{\omega}$

Frequency : No. of oscillations per unit time

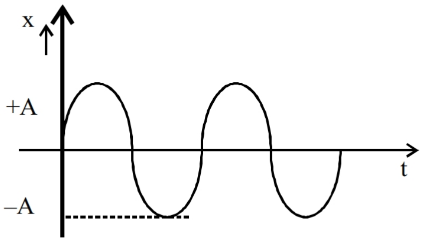
Angular fequency : $\omega = \frac{2\pi}{T} = 2\pi\nu$

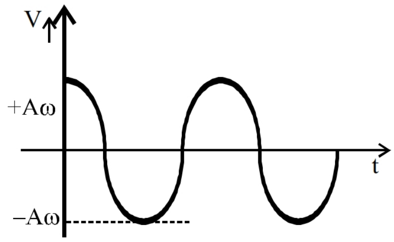
Velocity: $v = \frac{dx}{dt} = -A\omega\cos(\omega t + \phi) = \omega\sqrt{A^2 - x^2}$

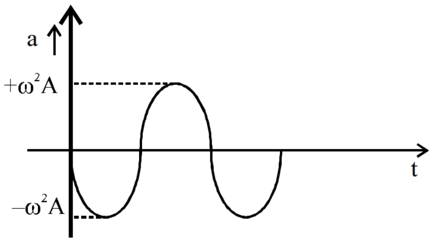
which varies periodically between the values $+\omega A$ and $-\omega A$

Acceleration: $a = \frac{dv}{dt} = -\omega^2 A\sin(\omega t + \phi) = -\omega^2 x$

and varies periodically between the values $+\omega^2 A$ and $-\omega^2 A$







Velocity and Acceleration as a Function of Displacement

By defintion, the acceleration of a body in S.H.M. is proportional to the displacement. i.e.

$$a = -\omega^2 x$$

$$a = \frac{dv}{dt} = -\omega^2 x$$

$$\Rightarrow \quad \frac{dv}{dx}.\frac{dx}{dt} = -\omega^2 x$$

$$\Rightarrow \quad v.\frac{dv}{dx} = -\omega^2 x \Rightarrow vdv = -\omega^2 x.dx$$

Integrating both sides

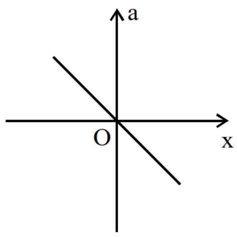
$$\int vdv = -\omega^2 \int x.dx$$

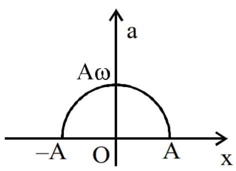
$$\frac{v^2}{2} = -\omega^2 \frac{x^2}{2} + c$$

when $x = 0$, $v = v_{max} = A\omega$

$$\therefore \quad C = \frac{A^2\omega^2}{2}$$

$$\therefore \quad \boxed{v = \omega\sqrt{A^2 - x^2}}$$





(a) Slop of the line $= -\omega^2$ (b) Variation of v with x is ellipse

KINETIC ENERGY

The kinetic energy of the particle is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi)$$

and using $x = A\sin(\omega t + \phi)$, we can also express K.E. as

$$K = \frac{1}{2}m\omega^2A^2\left(1 - \sin^2(\omega t + \phi)\right)$$

$$\boxed{K = \frac{1}{2}m\omega^2\left(A^2 - x^2\right)}$$

At $x = \pm A$, $K = K_{min} = 0$

At $x = 0$, $K = K_{max} = \frac{1}{2}m\omega^2A^2$

POTENTIAL ENERGY

To obtain the potential energy, we use the relation.

$$U = -\int_0^x F.dx = -\int_0^x (-Kx)dx$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

At $x = \pm A$, $U = U_{max} = \frac{1}{2}m\omega^2x^2$

At $x = 0$, $U = K_{min} = 0$

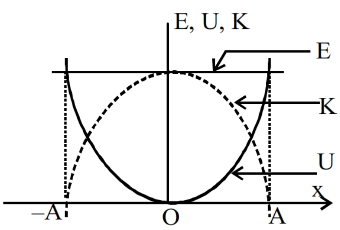
TOTAL ENERGY

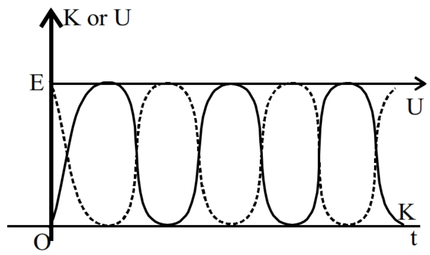
Total Energy E

$$E = K + U = \frac{1}{2}m\omega^2\left(A^2 - x^2\right) + \frac{1}{2}m\omega^2x^2$$

$$\boxed{E = \frac{1}{2}m\omega^2A^2}$$

which is a constant quantity. This was to be expected since the force is conservative.





TIME PERIOD AND FREQUENCY OF S.H.M.

Linear S.H.M.	Angular SHM
$F = -kx$	$\tau = -k\theta$
$ma = -kx$	$I\alpha = -k\theta$
$a = -\frac{k}{m}x$	$\alpha = -\frac{k}{I}\theta$
$= \omega^2 x$	$\alpha = -\omega^2\theta$
$\therefore \quad \omega = \sqrt{\frac{k}{m}}$	$\therefore \quad \omega = \sqrt{\frac{k}{I}}$
$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$	$\frac{2\pi}{T} = \sqrt{\frac{k}{I}}$
$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}}$	$\therefore \quad T = 2\pi\sqrt{\frac{I}{k}}$

General Formula therefore for Time period is $T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Force factor}}}$

SPRING MASS SYSTEM

(I) Series Combination

Two springs are said to be series when both are stretched with the same force F and the total displacement is the sum of individual deformation of each spring i.e.

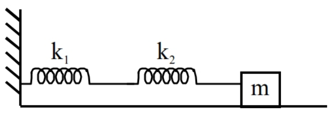
$$F = k_1x_1 = k_2x_2$$

$$x = x_1 + x_2$$

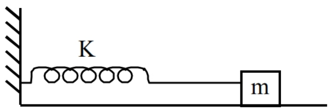
where x_1 and x_2 are deformations in the springs of constants k_1 and k_2 respectively.

Equivalent stiffness of the spring is given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$



and $\quad T = 2\pi\sqrt{\frac{m}{k}}$



(II) Parallel Combination

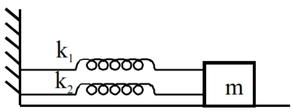
Two springs are said to be in parallel when both are stretched to the same deformation x and the total force F they exert on the block is equal to the sum of the individual forces. i.e.

$$x_1 = x_2 = x$$

$$F = F_1 + F_2 = (k_1 + k_2)x$$

$$\therefore \quad K = (k_1 + k_2)$$

$$T = 2\pi\sqrt{\frac{m}{K}}$$



RELATION BETWEEN STIFFNESS AND LENGTH OF THE SPRING

The stiffness K of a spring is inversely proportional to its length l.

$$K \propto \frac{1}{l}$$

If a spring of stiffness K and length l is cut into two parts of length l_1 and l_2, then the stiffness k_1 and k_2 of each part are given as

$$\frac{k_1}{k_2} = \frac{l_2}{l_1}$$

$$k_1 = \left(\frac{l_1 + l_2}{l_1}\right)K$$

$$k_2 = \left(\frac{l_1 + l_2}{l_2}\right)K$$

PENDULUMS

(i) Simple Pendulum: For small oscillations, the time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

The ime period is independent of mass.

(ii) If the time period of a simple pendulum is 2 seconds, it is called seconds pendulum.

(iii) If the length of the pendulum is large, g no longer remain vertical but will be directed towards the centre of the earth and then time period.

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{l} + \frac{1}{R}\right)}}$$

R = radius of the earth

(a) If $l << R$, $\frac{1}{l} >> \frac{1}{R}$ and $T = 2\pi\sqrt{\frac{l}{g}}$

(b) If $l >> R$, $\frac{1}{l} << \frac{1}{R}$ and $T = 2\pi\sqrt{\frac{l}{g}} = 84.6$ min

(iv) Time period of a simple pendulum depends on acceleration due to gravity. Then

$$T = 2\pi\sqrt{\frac{l}{|\vec{g}_{eff}|}} \qquad \text{where } \vec{g}_{eff} = \vec{g} - \vec{a}$$

(a) If a simple pendulum is in a carriage which is accelerating with acceleration $\vec{a}$, upwards, then

$$g_{eff} = g + a$$

$$T = 2\pi\sqrt{\frac{l}{a+g}}$$

(b) If the carriage is moving downwards.

$$g_{eff} = g - a$$

$$T = 2\pi\sqrt{\frac{l}{g-a}}$$

(c) If the carriage is in horizontal direction, then

$$g_{eff} = \sqrt{a^2 + g^2}$$

$$T = 2\pi\sqrt{\frac{l}{\sqrt{a^2+g^2}}}$$

(d) In a freely falling lift $g_{eff} = 0$, $T = \infty$, i..e. pendulum will not oscillate.

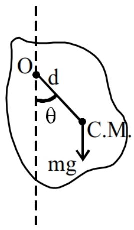
(e) If in addition to gravity, one additional force $\vec{F}$ (e.g. electrostatic force $\vec{F}_e$) is also acting on the bob, then in that case, $\vec{g}_{eff} = \vec{g} + \frac{\vec{F}}{m}$

PHYSICAL PENDULUM

A physical pendulum is an extended body pivoted about point O, which is at a distance d from its centre of mass. For small angular displacement θ,the restoring torque is given by

$$\tau = -mgd\sin\theta = -mgd\theta$$

Using Newton's second law



$$I\frac{d^2\theta}{dt^2} = -mgd.\theta$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = \frac{-mgd}{I}.\theta$$

Time period

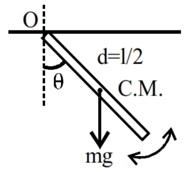
$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

Some Special Cases

(a) A rod of mass m and length l suspended about its its end.

Hence, $d = \frac{l}{2}$, $\quad I = \frac{ml^2}{3}$

$$\therefore \quad T = 2\pi\sqrt{\frac{2l}{3g}}$$



(b) Fig. shows a ring of mass m and radius R, pivoted at a point O on its periphery. It is free to rotate about an axis perpendicular to its plane.

Here, $\quad d = R$ and

$I = 2mR^2$

$$\therefore \quad T = 2\pi\sqrt{\frac{2R}{g}}$$